TCC Course "Weil Conjectures"

Exercises

The first two exercises give a direct proof of the Riemann Hypothesis for curves over finite fields. For this you should first read up on intersection theory on algebraic surfaces, e.g. in Hartshorne's "Algebraic Geometry", Section V.1.

1. Let C_1 and C_2 be smooth projective curves over an algebraically closed field k and X the surface $C_1 \times C_2$. Consider the divisors $\ell_1 = C_1 \times \mathrm{pt}$ and $\ell_2 = \mathrm{pt} \times C_2$ on X. Show that for any divisor D on X we have
$$(D.D) \leqslant 2(D.\ell_1)(D.\ell_2)$$
Hint: Apply the Hodge Index Theorem (Hartshorne V.1.9) to the divisors $\ell_1 + \ell_2$ and $D - (a\ell_1 + b\ell_2)$ for suitable integers a, b.

2. Let C be a smooth projective curve of genus g over a finite field $\mathbf{F}_q$ with q elements. Fix an algebraic closure $\bar{\mathbf{F}}_q$ of $\mathbf{F}_q$ and let $\bar{C} = C \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q$. Let $F\colon C \to C$ be the $\mathbf{F}_q$-Frobenius and $F\colon \bar{C} \to \bar{C}$ its base change to $\bar{\mathbf{F}}_q$. Let $X = \bar{C} \times \bar{C}$, $\Delta \subset X$ the diagonal and $\Gamma \subset X$ the graph of F.
   a) Show that the canonical divisor on X is linearly equivalent to $(2g-2)(\ell_1 + \ell_2)$, where the ℓ_i are defined as in Exercise 1.
   b) Using the adjunction formula (Hartshorne V.1.5) show that $(\Gamma.\Gamma) = q(2-2g)$ and $(\Delta.\Delta) = 2 - 2g$.
   c) Show that $|X(\mathbf{F}_q)| = (\Delta.\Gamma)$.
   d) By applying Exercise 1 to $D = a\Delta + b\Gamma$ for suitable a, b show that

(1)
$$||X(\mathbf{F}_q)| - (q+1)| \leqslant 2g\sqrt{q}.$$

   e) Write the zeta function of C as
$$\zeta_C(T) = \prod_{1=n}^{2g}(1 - \alpha_i T)/(1-T)(1-qT).$$
   Show that (1) implies $|\alpha_i| \leqslant \sqrt{q}$ for all i.

   Hint: Use the power series expansion
$$\sum_{i=1}^{2g} \alpha_i T/(1-\alpha_i T) = \sum_{j=1}^{\infty} a_j T^j.$$
   f) Using the functional equation for $\zeta_C(T)$ deduce that $|\alpha_i| = \sqrt{q}$ for all i.

3. We saw that the étale fundamental group of $\mathbf{A}^1_k$ over $k = \mathbf{C}$ is trivial. By contrast prove that for $k = \bar{\mathbf{F}}_p$, for every smooth projective curve C over k, there exists an open subset $U \subset C$ which admits a finite étale morphism $U \to \mathbf{A}^1_k$ as follows:
   a) Show that there exists a generically étale finite morphism $f\colon C \to \mathbf{P}^1_k$.
   b) For f as in a), take $V \subset \mathbf{A}^1_k$ open over which f is finite étale. Show that after possibly making V smaller, we have $\mathbf{A}^1_k \backslash V = \mathbf{F}_{p^n}$ for some $n \geqslant 1$ and that then $V \to \mathbf{A}^1 \backslash \{0\}$, $x \mapsto x^{p^n} - x$ is finite étale.
   c) Show that the morphism $\mathbf{A}^1_k \backslash \{0\} \to \mathbf{A}^1_k$, $x \mapsto x^p + 1/x$ is finite étale.