Basic question: $\mathbb{F}_q$ finite field

$X / \mathbb{F}_q$ variety

e.g.: $n \geq 1$: $f_i \in \mathbb{F}_q[x_0, \dots, x_n]$ homog. $i = 1, \dots, k$

$\leadsto X = Z(\{f_i\}) \subset \mathbb{P}^n_{/\mathbb{F}_q}$

For $n \geq 1$ $X(\mathbb{F}_{q^n})$ finite $\leadsto$ How do the $|X(\mathbb{F}_{q^n})|$ behave?

First step:

How to package the numbers $|X(\mathbb{F}_{q^n})|$?

Analogy with the Riemann zeta fct:

$$\zeta(s) = \prod_{\substack{p \text{ prime} \\ \text{in } \mathbb{Z}}} \frac{1}{1-p^{-s}} = \sum_{n\geq 1} n^{-s} \rightsquigarrow \text{anal. fct. } \mathrm{Re}(s)>1$$

- $\exists$ mero. continuation
- $\exists$ functional eq.
- Expect: non-trivial zeroes $\mathrm{Re}(s)=1/2$

Geom. int: $\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$

↑

set of closed pts of $\mathrm{Spec}(\mathbb{Z}) = X$

X scheme $\rightsquigarrow |X|$ set of cl. pts of underl. top. space

$x \in X \rightsquigarrow k(x)$ residue field

Here: $p = |k(x)|$

$\rightsquigarrow$ X var. / $\mathbb{F}_q$: $\zeta_X(s) := \prod_{x \in |X|} \frac{1}{1-|k(x)|^{-s}}$ $\rightsquigarrow$ Can show that this converges for $\mathrm{Re}(s) \gg 0$

- $\zeta_X(s) = \prod_{x \in |X|} \frac{1}{1-|k(x)|^{-s}}$
- look at it slightly diff:

  $x \in |X|$: $k(x) = \mathbb{F}_{q^{\deg(x)}}$
- $T = q^{-s}$ $\rightsquigarrow$ $\zeta_X(T) = \prod_{x \in |X|} \frac{1}{1-T^{\deg(x)}} \in \mathbb{Q}[[T]]$

  converges since there are only fin. many pts of bded deg.

$$f \in (T) \in \mathbb{Q}[[T]]$$

$$\rightsquigarrow \exp(f) = \sum_{n \geq 0} \frac{f^n}{n!}$$

$$f \in (T) \subset \mathbb{Q}[[T]] : \log(1+f) = \sum_{n \geq 1} (-1)^{n+1} \frac{f^n}{n}$$

$$\rightsquigarrow \log \zeta_X(T) = \log\left(\prod_{x \in |X|} \frac{1}{1-T^{\deg(x)}} \right) = \sum_{x \in |X|} \log\left(\frac{1}{1-T^{\deg(x)}} \right)$$

$$= \cancel{(-1)} \sum_{x \in |X|} \sum_{n \geq 1} \frac{T^{n\deg(x)}}{n}$$

$$\sum_{x\in|X|}\sum_{n\geq 1}\frac{T^{n\deg(x)}}{n}=\sum_{m\geq 1}T^m\sum_{\substack{x\in|X|\\ \deg(x)\mid m}}\frac{\deg(x)}{m}$$

$$=\sum_{n\geq 1}\frac{T^n}{n}\underbrace{\sum_{\substack{x\in|X|\\ \deg(x)\mid n}}\deg(x)}_{|X(\mathbb{F}_{q^n})|}$$

$y\in X(\mathbb{F}_{q^n})$.

$\longleftrightarrow \operatorname{Spec}(\mathbb{F}_{q^n})\to X$

$\longleftrightarrow x\in|X|$

$+ \ \underbrace{k(x)\to\mathbb{F}_{q^n}}_{\exists \deg(x) \text{ such homs}}$.

$$\Rightarrow \zeta_X(T) = \exp\left(\sum_{n \geq 1} |X(\mathbb{F}_{q^n})| \frac{T^n}{n} \right)$$

<u>Examples</u>: • $\mathbb{A}^n$: $|\mathbb{A}^n(\mathbb{F}_{q^m})| = q^{nm}$

$$Z_{\mathbb{A}^n}(T) = \exp\Big(\sum_{n\geq 1} q^{m} \frac{T^n}{n}\Big)$$

$$= \exp\Big(\log\Big(\frac{1}{1-q^n T}\Big)\Big) = \frac{1}{1-q^n T} \in \mathbb{Q}(T)$$

• $\mathbb{P}^n$: $\mathbb{P}^n = \mathbb{A}^n \amalg \mathbb{A}^{n-1} \amalg \dots \amalg \mathbb{A}^0$

$$Z_{X \amalg Y}(T) = Z_X(T)\, Z_Y(T)$$

$$\Rightarrow \zeta_{\mathbb{P}^n}(T) = \zeta_{\mathbb{A}^n}(T) \cdots \zeta_{\mathbb{A}^0}(T)$$

$$= \frac{1}{(1-q^nT)} \cdots \frac{1}{(1-T)}$$

Curves: X sm. proj. $/\mathbb{F}_q$

Recall: $\mathrm{Div}(X) = \left\{ \underbrace{\sum_i a_i [x_i]}_{} \;\middle|\; \begin{array}{l} a_i \in \mathbb{Z} \\ [x_i] \in |X| \end{array} \right\}$

$\deg(\ ) = \sum_i a_i \deg(x_i)$

$\mathrm{Pic}(X) = \{\text{line bundles}\}/\cong$ grp. by $\otimes$

$\mathrm{Div}(X_0) \twoheadrightarrow \mathrm{Pic}(X)$

$D \mapsto \mathcal{O}_X(D)$

$f \in K(X) \leftarrow$ function field of X

$\rightsquigarrow \quad |f| = \sum_{x \in |X|} \mathrm{ord}_x(f)\,[x] \in \mathrm{Div}(X)$

$\deg(|f|) = 0$

$\mathcal{O}_X(D) = \mathcal{O}_X(D') \Leftrightarrow D - D' = (f)$

$\rightsquigarrow$ get: $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$

Furthermore: $\mathrm{Div}^+(X) = \{ \sum a_i [x_i] \mid a_i \geq 0 \}$

$D \in \mathrm{Div}^+(X) \rightsquigarrow s \in H^0(\mathcal{O}_X(D), X)$ def. up to $\mathbb{F}_q^*$

s.t. get bij

$$\mathrm{Div}^+(X) \to \left\{ (L, s) \,\middle|\, \begin{array}{l} L \in \mathrm{Pic}(X) \\ s \in H^0(X, L) \end{array} \right\} / \mathbb{F}_q^*$$

<u>Riemann-Roch:</u> $\omega \in \mathrm{Pic}(X)$ = class of $\Omega^1_{X/\mathbb{F}_q}$

g = genus of X

For $L \in \mathrm{Pic}(X)$: $h^0(X, L) - h^0(X, \omega \otimes L^{-1})$
$= \deg(L) + 1 - g$

<u>Cor:</u> $\deg(L) > 2g-2 \Rightarrow h^0(X, L) = \deg(L) + 1 - g$

<u>Pf:</u> $\deg(\omega) = 2g-2 \Rightarrow \deg(\omega \otimes L^{-1}) < 0 \Rightarrow h^0(X, \omega \otimes L^{-1}) = 0$

<u>Lemma</u>: $\operatorname{Pic}^0(X) = \{ L \in \operatorname{Pic}(X) \mid \deg L = 0 \}$ is finite

<u>Pf</u>: Pick $n > 2g-2$

$L \in \operatorname{Pic}^n(X)$ ($\deg L = n$) $\rightsquigarrow h^0(L) > 0$

$\Rightarrow L = \mathcal{O}_X(D) \quad D = \sum a_i [x_i] \quad a_i \geq 0$

$\deg(D) = n$

$\rightsquigarrow \exists$ only fin. many such D for given n $\quad \operatorname{Pic}^0(X) \curvearrowright \operatorname{Pic}^n(X)$ simply trans.

Now: $\zeta_X(T) = \prod_{x \in |X|} (1 - T^{\deg(x)})^{-1}$

$$= \prod_{x \in |X|} \sum_{n \geq 0} T^{n \deg(x)} = \sum_{D \in \mathrm{Div}^+(X)} T^{\deg(D)}$$

$D \in \mathrm{Div}^+(X)$
$\leftrightarrow (L, s)$ $L \in \mathrm{Pic}(X)$
$0 \neq s \in H^0(L)/\mathbb{F}_q^*$

$$= \sum_{\substack{L \in \mathrm{Pic}(X) \\ h^0(L) > 0}} \left| H^0(L)^{\setminus\{0\}} / \mathbb{F}_q^* \right| T^{\deg(L)}$$

$$= \sum_{\substack{L \in \mathrm{Pic}(X) \\ \deg(L) \geq 0}} \frac{q^{h^0(L)} - 1}{q-1} T^{\deg(L)}$$

$$= \sum_{0 \leq \deg(L) \leq 2g-2} \cdots + \underbrace{\sum_{\deg(L) > 2g-2}}$$

$h^0(L) = \deg(L) + 1 - g$

$$= \sum_{2g-2 < \deg(L)} \frac{q^{\deg(L)+1-g} - 1}{q-1} T^{\deg(L)}$$

$$\sum_{2g-2<\deg(L)} \frac{q^{\deg(L)+1-g}-1}{q-1} T^{\deg(L)}$$

$$= |\mathrm{Pic}^0(X)| \sum_{2g-2<n} \frac{q^{n+1-g}-1}{q-1} T^n$$

Can prove: $\exists L$ of deg 1 $= \ldots.$ rational

(Or assume $X(\mathbb{F}_q) \neq \emptyset$) $\Rightarrow Z_X(T) = \dfrac{h(T)}{(1-T)(1-qT)}$ $\deg h(T) \leq 2g$, $h \in \mathbb{Q}[T]$

Func. eq:

Thm $\zeta_X(q^{-1}T^{-1}) = q^{1-g}\, T^{2-2g}\, \zeta_X(T)$

Idea of: $L \mapsto \omega \otimes L^{-1}$

bijection on $\{L \mid 0 \le \deg(L) \le 2g-2\}$

Equivalently: $\zeta_X(T) = \frac{h(X)}{(1-T)(1-qT)}$ $h(X) = \prod_{i=1}^{2g} (1-\alpha_i T)$ $\alpha_i \in \mathbb{C}$

$\rightsquigarrow$ the α_i show up in pairs $\alpha_i \alpha_{i'} = q$ $(\Rightarrow \deg(h) = 2g)$

$\rightsquigarrow$ Have pairs $\alpha_i \cdot \alpha_{i'} = q$

$\alpha_i, \overline{\alpha_i}$

$\rightsquigarrow$ $q/\alpha_i \overset{?}{=} \overline{\alpha_i}$?

<u>Thm</u>: $\forall i : |\alpha_i| = q^{1/2}$

<u>Rmk</u>: Under $T = q^{-s}$ this means that all zeroes in s have $\mathrm{Re}(s) = 1/2$

Idea of pf: Look at $\overline{X} := X_{\overline{\mathbb{F}_q}}$

$S = \overline{X} \times \overline{X}$ surface

Have: $F : X \to X$ Frobenius $x \mapsto x^q$

$\rightsquigarrow F : \overline{X} \to \overline{X}$ $X(\mathbb{F}_q) =$ fixed points of F^n

$\Gamma \subset S$ graph of F

$\Delta =$ diagonal $: \overline{X} \to \overline{X} \times \overline{X}$ $\Delta \cap \Gamma$ $(n=1)$

$\rightsquigarrow$ Use intersection theory on S

(Hartshorne: Ex V.1.10
C5.7)

Rmk: $|\alpha_i| = q^{1/2} \Rightarrow \big|\, |X(\mathbb{F}_{q^n})| - 1 - q^n \big| \le 2g\, q^{n/2}$

Pf: $N_n = |X(\mathbb{F}_{q^n})|$

$$\exp\Big(\sum_{n\geq 1} N_n \frac{T^n}{n}\Big) = \frac{\prod_i (1-\alpha_i T)}{(1-T)(1-qT)}$$

$$\log(\quad) = \sum_i \log(1-\alpha_i T) - \log(1-T) - \log(1-qT)$$

$$\sum_{i=1}^{2g} \log(1-\alpha_i T) - \log(1-T) - \log(1-qT)$$

$$= \sum_{i=1}^{2g} \sum_{n} \left(\alpha_i^n - 1 - q^n\right) \frac{T^n}{n}$$

$$\Rightarrow \quad N_n = -\sum_{i=1}^{2g} \alpha_i^n + 1 + q^n \qquad \square$$

Thm

X sm. proj. var $/\mathbb{F}_q$ $d = \dim X$

(1) $Z_X(T) \in \mathbb{Q}(T)$

(2) $$Z_X(T) = \frac{P_1(T)P_3(T)\ldots P_{2d-1}(T)}{P_0(T)P_2(T)\ldots P_{2d}(T)}$$ s.t. $P_i \in \mathbb{Z}[T]$, cst. 1, all cplx roots of P_i have $|\cdot| = q^{i/2}$

$P_0(T) = 1-T$ $P_{2d}(T) = 1-q^dT$

(3) (Func. eq):

$\exists\, \chi \in \mathbb{Z}^{\geq 0}$, $\varepsilon \in \{\pm 1\}$ s.t.:

$$Z_X\left(\frac{1}{q^d T}\right) = \varepsilon\, q^{d\chi/2}\, T^{\chi}\, Z_X(T)$$

(4) If "X lifts to char. 0":

$\exists F$ local field of char. 0 (e.g. $\mathbb{Q}_p$) $\mathcal{O}_F \subset F$ val. ring

$\mathbb{F}_q = \mathcal{O}_F/\mathfrak{m}_F$

$\exists\, \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ smooth proj scheme / $\mathcal{O}_F$

s.t. $\mathcal{X}_{\mathbb{F}_q} = X$

Choose $F \hookrightarrow \mathbb{C}$

$\rightsquigarrow$ $\mathcal{X}(\mathbb{C})$ smooth mfld

$\rightarrow \deg P_i = \dim_{\mathbb{Q}} H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q})$ (top. sing. coh.

Applications:

(1) $$|X(\mathbb{F}_{q^n})| = 1 + q^{nd} + O\left(q^{n(d-1/2)}\right)$$

(2) $$f(z) = e^{2\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)^{24}$$

$$= \sum_{n \geq 0} \underbrace{\tau(n)}_{\in \mathbb{Z}} e^{2\pi i n z}$$

$\forall \varepsilon > 0$ $$|\tau(n)| = O_\varepsilon\left(n^{11/2+\varepsilon}\right)$$

(3) (Batyrev)

Thm: X, Y sm. proj. var. $/\mathbb{C}$

s.t.: · $\Omega^1_{X/\mathbb{C}}, \Omega^1_{Y/\mathbb{C}}$ are trivial v.b

· X and Y are birational

$\Rightarrow$ X and Y have the same Betti #