How to define a "top." coh. theory for varieties?

X "nice" top space, e.g. manifold

$\rightsquigarrow H^i(X, \mathbb{Q})$ top. coh.

1) Sing. coh: look at cont maps

$\underset{\substack{\uparrow \\ \text{st. n-simplex}}}{\Delta^n} \to X$ $\rightsquigarrow$ no good for var. with Zar. top

2) Sheaf coh:

Take $\mathbb{Q} :=$ cst sheaf on X with group $\mathbb{Q}$

$\rightsquigarrow H^i(X, \mathbb{Q})$ sheaf coh.

gives the same as sing coh for good X

For X irr. var. $\rightsquigarrow$ take cst $\mathcal{F}$ with val. gr $\mathbb{Q}$ w.r.t Zar top

$\forall U, U' \subset X$: $U \cap U' \neq \emptyset$

$$\begin{tikzcd} \mathbb{Q} \arrow[r, equal] & \mathcal{F}(U) \arrow[d, "\cong"] & \mathcal{F}(U') = \mathbb{Q} \arrow[ld, "\cong"'] \\ \mathbb{Q} \arrow[r, equal] & \mathcal{F}(U \cap U') & \end{tikzcd}$$

$\Rightarrow H^i(X, \mathcal{F}) = 0 \quad \forall i > 0$

Grothendieck's idea: replace open immersions $U \hookrightarrow X$ by a bigger class of morphisms $U \to X$ to get a better class of sheaves.

$\rightsquigarrow$ Étale morphisms:

$/\mathbb{C}$: for smooth var. $/\mathbb{C}$, an étale morphism $U \to X$ is a quasi-finite mor, which is locally for the analytic top. an iso.

From now on: all schemes are locally noeth, all morphisms loc. of finite type

<u>Def</u>: $f: X \to Y$ mor. is:

- unramified if $\forall x \in X$: $y = f(x)$

$$k(y) = \mathcal{O}_{Y,y}/m_y \to \mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x}$$ is sep. ext. of fields.

- étale if it is unr. + flat

<u>E.g.</u>: $f: \mathbb{A}^1_k \to \mathbb{A}^1_k$, $x \mapsto x^n$, is étale restricted to $\mathbb{G}_m$ for $(n, \operatorname{char} k) = 1$

At 0: $k[x]_{(x)} \to k[x]_{(x)}$, $x \mapsto x^n$ $\supset (x^n)$ not max. ideal

Equivalenty:

- $f: X \to Y$ unr $\iff \Omega^1_{X/Y} = 0$
- $f: X \to Y$ étale $\iff$

$\forall x \in X \quad \exists$:

$$\begin{array}{ccccc} x \in \mathrm{spec}(B) = U & \subset & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) = V & \subset & Y \end{array}$$

s.t.: $B = A[T_1, \dots, T_n] / (P_1, \dots, P_n)$ s.t. $\det \left(\partial P_i / \partial T_j \right)_{i,j=1,\dots,n}$ is invertible in B

Fact: The image of an étale mor. is always open.

Ex: $k \hookrightarrow k'$ sep field ext is étale, open immersions are flat
k field

$\rightsquigarrow$ sheaf theory:

X scheme

Def: An étale sheaf of ab. grps. consists of:

- For every étale mor. $U \to X$ an ab grp. $F(U)$
- For all $\begin{array}{ccc} U & \longrightarrow & U' \\ & {\scriptstyle \text{ét.}}\searrow \quad \swarrow{\scriptstyle \text{ét.}} & \\ & X & \end{array}$ (in this case $U \to U'$ is aut. étale)

$\rho^{U'}_{U} : F(U') \to F(U)$

s.t.: $\cdot$ $\rho^{U}_{U} = \mathrm{id}_{F(U)}$, $\forall$ $\begin{array}{c} U \to U' \to U'' \\ {\scriptstyle \text{ét}} \searrow \downarrow \swarrow \\ X \end{array}$ $\rho^{U'}_{U} \circ \rho^{U''}_{U'} = \rho^{U''}_{U}$

· Sheaf axiom: $\forall\, (U_i \xrightarrow{ét} X)_{i\in I}$ s.t. $X = \bigcup_{i\in I} f_i(U_i)$

$$0 \to F(X) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_X U_j)$$

$$\begin{tikzcd} & U_i \times_X U_j \arrow[dl, "et"'] \arrow[dr, "et"] & \\ U_i \arrow[dr] & & U_j \arrow[dl] \\ & X & \end{tikzcd}$$

restr. along the two proj.

is exact

$\rightsquigarrow$ Get an ab. cat. of ét sheaves Sh_X^{et} on X

$+\ \Gamma : Sh_X^{et} \to (\text{Ab. grps})$ $\rightsquigarrow$ can form derived functors

$F \mapsto F(X)$ $\qquad$ $F \mapsto H^i_{et}(X, F)$

left exact $\qquad$ et. coh. grps

- Unfortunately, this coh. theory is only well-beh. for torsion sheaves

  Eg: $H^2_{et}(\mathbb{P}^1, \mathbb{Q}) = 0$ $(/k=\bar{k})$
  cst. sheaf

- But can compute:
  X sm. proj. conn curve $/k=\bar{k}$ g=genus of X
  $(n, \mathrm{char}(k))=1$

  $H^0_{et}(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ $H^1_{et}(X, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$

  $H^2_{et}(X, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$
  cst sheaf

Work $/k$

To get a theory with char. 0 coeff.

pick ℓ s.t $(\ell, \mathrm{char}\, k) = 1$

$$\text{For } X/k \rightsquigarrow H^i_{et}(X, \mathbb{Q}_\ell) := \varprojlim H^i_{et}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

($\mathbb{Z}/\ell^n\mathbb{Z} \circlearrowleft$)

Fact: This gives a Weil coh. theory. for $k = \overline{\mathbb{F}_p}$, $p \neq \ell$

Want to consider coh. of more gen. objects than $\mathbb{Q}_\ell$, e.g. to consider how coh. varies in a family of alg. var.

Top motivation

X, S smooth mflds $X \xrightarrow{f} S$ smooth surj. submersion proper

look at this as a fam. of sm. proj. mflds

Ehresmann: f is a loc. triv. fibration: $f^{-1}(U) \cong X_s \times U \to X$ $\quad X_s = f^{-1}(s)$

$$\begin{array}{ccc} f^{-1}(U) \cong X_s \times U & \to & X \\ \downarrow & & \downarrow f \\ U & \subset & S \end{array}$$

$\rightsquigarrow$ The coh. groups $H^i(X_s, \mathbb{Q})$ fit together in a sheaf on S of $\mathbb{Q}$-vs which is loc. cst

$\underbrace{\text{sheaf}}_{R^i f_* \mathbb{Q}}$

$\rightsquigarrow$ the class of loc. cst sheaves of $\mathbb{Q}$-v.s. gives a "good" class of coeff. objects, invariant under desired pushforward.

$\rightsquigarrow$ Want to do the same for étale sheaves

<u>Def</u>: X scheme. The cat. of lisse / constructible $\mathbb{Q}_\ell$-sheaves is given as follows:

<u>Objects</u>: $(F_n)_{n\geq 1}$ $F_n \in Sh_X^{et}$ sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-mod.

+ $F_{n+1}/\ell^n F_{n+1} \cong F_n$ for each $n\geq 1$

s.t: Each F_n is:

lisse: locally cst: $\exists (U_i \xrightarrow{f_i} X)_{i\in I}$ ét. cov.

s.t for each i: $\exists\, r \geq 1$ $f_i^* F_n \cong \left(\mathbb{Z}/\ell^n\mathbb{Z}\right)^{\oplus r}$

cst' sheaf

constr.: $\exists$ decomp. $X = \coprod X^i$ into loc. cl. subschemes X^i

s.t $F_n|_{X^i}$ is loc. cst.

(the dec. can depend on n)

<u>Morphisms:</u> $F = (F_n)$ $G = (G_n)$

$\mathrm{Hom}(F, G) = \left\{ \begin{matrix} \text{compat families of} \\ \text{mor } F_n \to G_n \end{matrix} \right\} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

- $F = (F_n)_{n\geq 1}$ is a constr. $\mathbb{Q}_\ell$-sheaf

  $\rightsquigarrow$ $H^i_{et}(X,F) := \varprojlim_n H^i_{et}(X, F_n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$
- These cat are ab. cat.

Let: $f: X \to S$ proper morph.
For F constr. $\mathbb{Q}_\ell$-sheaf on X get
$f_* F$ a constr. $\mathbb{Q}_\ell$-sheaf on S.
$+$ constr. $\mathbb{Q}_\ell$-sheaves $R^i f_* F$

and: Let L be a sep. closed field
+ $p: \mathrm{Spec}(L) \to S$

$$p^* R^i f_* F \cong H^i_{et}(X_s, F|_{X_s})$$

↑
constr. $\mathbb{Q}_\ell$-sheaf on $\mathrm{Spec}(L)$
$\longleftrightarrow$ f. dim $\mathbb{Q}_\ell$-v.s.

Furthermore: If f is in addition smooth and F is lisse, then the $R^i f_* F$ are also lisse.

Comparison to top. coh:

$X / \mathbb{C}$ smooth proper

top. coh.

$\rightsquigarrow H^i_{et}(X, \mathbb{Q}_\ell) \cong H^i(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$

R local ring, $R/m = k = \bar{k}$ $\quad R \hookrightarrow \mathbb{C}$

By above and smooth proper $X \to \mathbb{C}$

$\mathcal{X} \xrightarrow{f} \operatorname{Spec}(R)$ sm. proper mor. $\quad i : \operatorname{Spec}(k) \to \operatorname{Spec}(R)$, $j : \operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(R)$

$\rightsquigarrow R^i f_* \mathbb{Q}_\ell$ lisse $\mathbb{Q}_\ell$-sheaf on $\operatorname{Spec}(R)$

$i^* R^i f_* \mathbb{Q}_\ell \cong H^i_{et}(\mathcal{X}_k, \mathbb{Q}_\ell)$, $j^* R^i f_* \mathbb{Q}_\ell \cong H^i_{et}(\mathcal{X}_{\mathbb{C}}, \mathbb{Q}_\ell)$ have the same dim.