Cohomology of Lefschetz pencils:

Top. situation: $\begin{matrix} X \\ \downarrow f \\ \mathbb{P}^1 \end{matrix}$ ← hol. map, proper, only critical points are / $x_1, \ldots, x_n \in \mathbb{P}^1$ each fiber has at most one sing. pt, a non-deg. crit. pt

Hessian $\neq 0$
$\longleftrightarrow$ ord. double pts

dim. of the fibers of $f = 2m+1 = n$

Fix i : $D_\varepsilon(x_i)$ ε-ball

For ε small: $f^{-1}(x_i) \subset f^{-1}(D_\varepsilon(x_i))$ $\exists$ def. retract R

$\rightsquigarrow$ For $s \in D_\varepsilon(x_i) \setminus \{x_i\}$: $H_*(f^{-1}(s)) \to H_*(f^{-1}(D_\varepsilon(x_i))) \xrightarrow{R} H_*(f^{-1}(x_i))$

specialization map

+ monodromy operator : $T: H_*(f^{-1}(s)) \to H_*(f^{-1}(s))$

monodromy along some loop around x_i in $D_\varepsilon(x_i)$

$\rightsquigarrow$ Thm (Picard-Lefschetz formulas)

- spec. maps are an iso on H_i for $i \neq n, n+1$
- Exact seq:

$$0 \to H_{n+1}(f^{-1}(s)) \to H_{n+1}(f^{-1}(x_i)) \to \mathbb{Q}$$
$$1 \mapsto \delta$$
$$\to H_n(f^{-1}(s)) \to H_n(f^{-1}(x_i)) \to 0$$

δ "vanishing cycle"

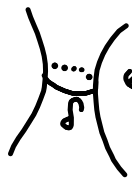
$T = \mathrm{id}$ on H_i for $i \neq n$

$\gamma \in H_n(f^{-1}(s))$: $T_n(\gamma) = \gamma \pm \langle \gamma, \delta \rangle \delta$

$H_n(f^{-1}(s)) \times H_n(f^{-1}(s)) \to \mathbb{Q}$ P.D

Picture: $X = \{ x^2 + y^2 = a \} \mapsto a \in \mathbb{P}^1$
$x, y, a \in \mathbb{C}$

Fiber / $a \in \mathbb{R}^{>0}$:



← radius goes to 0 as $a \to 0$

Algebro-geom. setting:

<u>Def</u>: · A local ring R is Henselian if it satisfies Hensel's Lemma:

For any monic $P \in R[T]$, any factorization of $\bar{P} \in (R/m)[T]$ into monic coprime polynomials lifts to R (uniquely?)

· R is strictly Hens. if it Hens. + R/m is sep. cl.

<u>Fact</u>: R is str. Henselian $\Leftrightarrow$ Every étale covering of Spec(R) is $\cong \coprod_{i \in I}$ Spec(R)

- E.g every complete local ring is Henselian
- R is local ring $\rightsquigarrow$ $R \hookrightarrow \varinjlim_{R \hookrightarrow R'} R' =: R^{sh}$

  strict Henselization of R; $\operatorname{Spec}(R') \to \operatorname{Spec}(R)$ étale, conn.; R^{sh} strict hens. ring

Fix: R str. hens. ring, a dvr, K = quot. field, ℓ a prime inv. in R, $k = R/m$ $(k = \bar{k})$

Tate twists: $\mathbb{Q}_\ell(1) := (\varprojlim \mu_{\ell^n}(R)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ a 1-dim $\mathbb{Q}_\ell$-v.s.

$\mathbb{Q}_\ell(-1) := \mathbb{Q}_\ell(1)^\vee$, $n \geq 1$ $\mathbb{Q}_\ell(n) = \mathbb{Q}_\ell(1)^{\otimes n}$, $n \leq -1$: $\mathbb{Q}_\ell(n) = \mathbb{Q}_\ell(-n)^\vee$

- M is $\mathbb{Q}_\ell$-v.s, $n \in \mathbb{Z}$ $\rightsquigarrow$ $M(n) := M \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(n)$

- $\mathrm{Gal}(K^s / K) =: G$ (sep. cl.) Choose $\pi \in R$ unif.

For $n \geq 1$: Choose π^{1/ℓ^n} ℓ^n-th root of π

$\rightsquigarrow G \to \mu_{\ell^n}(R)$

$\sigma \mapsto \sigma(\pi^{1/\ell^n}) / \pi^{1/\ell^n}$ ind. of choices

$\overset{\varprojlim}{\rightsquigarrow}$ $G \to \mathbb{Q}_\ell(1)$

Thm (local Picard–Lefschetz)

$S = \mathrm{Spec}(R)$ $\quad X \xrightarrow{f} S$ proper + flat $\quad \eta : \mathrm{Spec}(K^s) \to \mathrm{Spec}(R)$

fiber dim of f : $n = 2m+1$

f has single sing. pt a which $\quad s : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$

lies in X_s and is an ord. double pt.

$\rightsquigarrow$ Etale loc. around a, $X \xrightarrow{f} S$ looks like

$$\mathrm{Spec}\left(R[X_0, \ldots, X_n] \Big/ \left(\sum_{i=0}^{m} X_i X_{i+1+m} + \pi^\mu \right) \right) \to \mathrm{Spec}(R)$$

- Have: sp. maps $H^k(X_s, \mathbb{Q}_\ell) \to H^k(X_{\eta_1}, \mathbb{Q}_\ell)$

  + an action of G on $H^k(X_{\eta_1}, \mathbb{Q}_\ell)$

<u>Thm</u>:
- $H^k(X_s, \mathbb{Q}_\ell) \xrightarrow{\cong} H^k(X_{\eta_1}, \mathbb{Q}_\ell)$ for $k \neq n, n+1$
- $\exists\, \delta \in H^n(X_{\eta_1}, \mathbb{Q}_\ell)(m)$ "van. cycle", well-def up to sign

  $\exists\, \delta^* \in H^{n+1}(X_s, \mathbb{Q}_\ell)(n-m)$ "covan. cycle" s.t.

$$0 \to H^n(X_s, \mathbb{Q}_\ell) \to H^n(X_{\eta_1}, \mathbb{Q}_\ell) \xrightarrow{x \mapsto \langle x, \delta \rangle} \mathbb{Q}_\ell(m-n)$$
$$\xrightarrow{\lambda \mapsto \lambda \cdot \delta^*} H^{n+1}(X_s, \mathbb{Q}_\ell) \to H^{n+1}(X_{\eta_1}, \mathbb{Q}_\ell) \to 0$$

where : $H^n(X_{\bar\eta}, \mathbb{Q}_\ell) \times H^n(X_{\bar\eta}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-n)$
P.D. pairing

- G acts triv. on $H^k(X_{\bar\eta}, \mathbb{Q}_\ell)$ for $k \neq n$, and on $H^n(X_{\bar\eta}, \mathbb{Q}_\ell)$ by

$$\begin{array}{l}\sigma \in G \\ x \in H^n(X_{\bar\eta}, \mathbb{Q}_\ell)\end{array} \quad \sigma(x) = x + \underbrace{(-1)^m \chi(\sigma)^m}_{\text{twist by } 1+m-n+m=0} \langle x, \delta \rangle \delta$$

Global picture: $\quad X / k = \bar{k}$ sm. proj.

$X \hookrightarrow \mathbb{P}^r \qquad \dim X = n+1 \qquad n = 2m+1$

$$\begin{tikzcd} X & \tilde{X} \arrow[l] \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & \{(x,H) \in X \times \check{\mathbb{P}}^r \mid x \in H\} \arrow[d] \\ & D = \mathbb{P}^1 \arrow[r, hook] & \check{\mathbb{P}}^r = \{H \subset \mathbb{P}^r \text{ hyperpl.}\} \end{tikzcd}$$

f flat + proper, smooth outside $A \subset D$ finite

$\rightsquigarrow R^i f_* \mathbb{Q}_\ell$ constr. $\mathbb{Q}_\ell$-sheaves on D, lisse on $D \setminus A$

$K(D)$ func. field of D

$\Omega = K(D)^s \rightsquigarrow \mathrm{Spec}(\Omega) \xrightarrow{\omega} D \setminus A$ gen. pt

$\rightsquigarrow \pi_1(D \setminus A, \omega)$ et. fund'. gr.

$\left\{ \begin{array}{l} \text{lisse } \mathbb{Q}_\ell\text{-sheaves} \\ \text{on } D \setminus A \end{array} \right\} \longleftrightarrow \left\{ \text{cont. } \mathbb{Q}_\ell\text{-rep's of } \pi_1(D \setminus A, \omega) \right\}$

$F \mapsto F_\omega$

$\mathcal{G}(V) \leftarrow V$ choose

$s \in A \qquad \mathcal{O}_{D,s} \hookrightarrow R(s) = \mathcal{O}_{D,s}^{st} \hookrightarrow \Omega \rightsquigarrow \tilde{\omega} : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(R(s)) = D(s)$

str. hens.

$K(s)$ = quot. field of $R(s)$ $\qquad p = \mathrm{char}\, k$

$$\rightsquigarrow \pi_1(D(s)\setminus\{s\}, \bar{\omega}) = \mathrm{Gal}(\Omega / K(s)) \to \pi_1(D \setminus A, \omega)$$

$$\downarrow \rho$$

$\rho : \sigma \mapsto \rho(\sigma)$: $\qquad \varprojlim_{p \nmid r} \mu_r(k) = \varprojlim_{p \nmid r} \mu_r(R(s))$

$\sigma(\sqrt[r]{\pi}) = \rho(\sigma)\sqrt[r]{\pi}$ $\qquad \parallel$

$I_s = \ker \rho$ $\qquad \hat{\mathbb{Z}}^{(p)}(1)$ $\qquad$ 1. dim free mod

$$\hat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell$$

$$\pi_1^t(D\setminus A,\omega) := \pi_1(D\setminus A,\omega) \big/ \langle I_s \rangle_{s\in A}$$ ← norm. subgr.

tame fund. group.

$$\begin{tikzcd} \pi_1(D(s)\setminus\{s\},\bar{\omega}) \arrow[r] \arrow[d, "\rho"] & \pi_1(D\setminus A,\omega) \arrow[d] \\ \hat{\Sigma}^{(p)}(1) \arrow[r, dashed, "\gamma_s"] & \pi_1^t(D\setminus A,\omega) \end{tikzcd}$$

images of these for $s\in A$ generat RHS

<u>Def:</u> Call a $\pi_1(D\setminus A, \omega)$-rep <u>tame</u> if it factors through $\pi_1^t(\ ,\ )$

<u>Rmk:</u> $\pi_1(D, \omega) = \{1\} \Rightarrow$ Every lisse $\mathbb{Q}_\ell$-sheaf on D is constant.

<u>Thm:</u> $V = (R^n f_* \mathbb{Q}_\ell|_\omega = H^n(\bar{X}_\omega, \mathbb{Q}_\ell) \circlearrowleft \pi_1(D\setminus A, \omega)$

- The sheaves $R^i f_* \mathbb{Q}_\ell$ for $i \neq n, n+1$ are const. on D
- For each $s \in A$ get $\delta_s \in V(m)$ well-def up to sign up to conj by $\pi_1(D\setminus A, \omega)$ and $\delta_s' \in (R^n f_* \mathbb{Q}_\ell)_s(?-m)$

+ have ex. seq:

$$0 \to (R^n f_* \mathbb{Q}_\ell)_s \to (R^n f_* \mathbb{Q}_\ell)_w \xrightarrow{x \mapsto \langle x, \delta_s \rangle} \mathbb{Q}_\ell(m-n)$$

$$\xrightarrow{\lambda \mapsto \lambda \delta_s^*} (R^{n+1} f_* \mathbb{Q}_\ell)_s \to (R^{n+1} f_* \mathbb{Q}_\ell)_w \to 0$$

- The sheaves $R^n f_* \mathbb{Q}_\ell$, $R^{n+1} f_* \mathbb{Q}_\ell$ are lisse on $D \setminus A$

$\pi_1(D \setminus A, w)$ acts trivially on $(R^{n+1} f_* \mathbb{Q}_\ell)_w$ and tamely on $(R^n f_* \mathbb{Q}_\ell)_w = V$

For $x \in V$, $u \in \hat{\mathbb{Z}}^{(p)}(1)$: $\qquad \gamma_s : \hat{\mathbb{Z}}^{(p)}(1) \to \pi_1^t(D \setminus A, w)$

$$\gamma_s(u) x = x - (-1)^m \bar{u} \langle x, \delta_s \rangle \delta_s \qquad u \mapsto \bar{u} \in \mathbb{Z}_\ell$$

<u>Def</u>: $E := \sum_{\substack{s \in A \\ \sigma \in \pi_1^+(D \setminus A, \omega)}} \mathbb{Q}_\ell(-m) \cdot \delta_s \subset V$ space of van. cycles

$V \times V \to \mathbb{Q}_\ell(-n)$ P.D non-deg. alt. pairing

$\rightsquigarrow \pi_1^+(D \setminus A, \omega) \to Sp(V)$ ← sympl. group

<u>Can show</u>: The δ_s for var. s are conj. under $\pi_1^+(D \setminus A, \omega)$ up to sign,
in part if one $\delta_s = 0$, they all are

Get: $E / E \cap E^\perp$ has again non-deg. pairing, has an action of π_1^+, under which is abs. irr.

$j : D \setminus A \to D \rightsquigarrow j_* :$ constr. sheaves on $D\setminus A$ $\to$ constr. sheaves on D

Thm Case: $E = 0$: $R^n f_* \mathbb{Q}_\ell$ is cst on D

$$0 \to \underbrace{\bigoplus_{s\in A} \mathbb{Q}_\ell(m-n)_s}_{\text{skyscraper sheaf at } s} \to R^{n+1} f_* \mathbb{Q}_\ell \to j_*(j^*(R^{n+1} f_* \mathbb{Q}_\ell)) \to 0$$

$*$

Case $E \neq 0$: $R^{n+1} f_* \mathbb{Q}_\ell$ is cst on D

$R^n f_* \mathbb{Q}_\ell = j_*(j^*(R^n f_* \mathbb{Q}_\ell)) = j_* \mathcal{G}(V)$

(a) $E \subset E^\perp$: Have an exact seq:

$$0 \to j_*(\mathcal{G}(E)) \to R^n f_* \mathbb{Q}_\ell \to j_*(\mathcal{G}(V/E^\perp)) \to \bigoplus_{s\in A} \mathbb{Q}_\ell(m-n)_s \to 0$$

$*$ $*$

(b) $E \not\subset E^\perp$:

$$0 \to j_*(\mathcal{G}(E)) \to R^1 f_* \mathbb{Q}_\ell \to j_*(\mathcal{G}(\overset{*}{V}/E)) \to 0$$

$$0 \to j_*(\underset{*}{\mathcal{G}(E \cap E^\perp)}) \to j_*(\mathcal{G}(E)) \to j_*(\underset{*}{\mathcal{G}(E/_{E\cap E^\perp})}) \to 0$$

· Sheaves marked $*$ are cst on D

Rmk: If $X / \mathbb{F}_q$ $k = \overline{\mathbb{F}_q}$

⇝ after repl. $\mathbb{F}_q$ by a finite ext. $\tilde{X} \xrightarrow{f} \mathbb{P}^1$, the sing. pts

all sheaves, exact seq. are def / $\mathbb{F}_q$.