Coh <u>with cpct support</u>:

Top. sit: X mfld, F loc. cst sheaf of $\mathbb{Q}$-v.s.
$d = \dim X$

$$\Gamma_c(X,F) = \{ s \in \Gamma(X,F) \mid \operatorname{supp}(s) \text{ cpct} \}$$

supp(s) = $x \in X : s_{(x)} \neq 0 \in \mathcal{O}_{X,x}$

left exact functor $\rightsquigarrow$ $H^i_c(X,F)$ der. functors f. dim

If X is oriented get $H^i_c(X,\mathbb{Q}) \times H^{d-i}(X,\mathbb{Q}) \xrightarrow{\text{P.D duality}} H^d_c(X,F) \cong \mathbb{Q}$

Diff picture:

Look at $X \underset{\text{open}}{\hookrightarrow} \overline{X}$ ← cpct mfld

F sheaf on $X \rightsquigarrow$ extend by 0 to $\overline{X}$:

$$U \subset \overline{X}: \quad \overline{F}(U) = \begin{cases} F(U) & U \subset X \\ 0 & U \not\subset X \end{cases}$$

$$\rightsquigarrow H^i_c(X, F) \cong H^i(\overline{X}, \overline{F})$$

<u>Alg setting:</u> $X \overset{j}{\hookrightarrow} Y$ open imm of schemes

ℓ prime invertible on X on Y

$j_! : \left\{ \begin{array}{c} \text{etale sheaves} \\ \text{on } X \end{array} \right\} \to \left\{ \begin{array}{c} \text{etale sh.} \\ \text{of } Y \end{array} \right\}$

"ext. by 0"

$\left\{ \begin{array}{c} \text{constr. } \mathbb{Q}_\ell\text{-} \\ \text{sheaves} \\ \text{on } X \end{array} \right\} \to \left\{ \begin{array}{c} \text{constr. } \mathbb{Q}_\ell\text{-sh} \\ \text{on } Y \end{array} \right\}$

<u>Def:</u> $X \xrightarrow{f} S$ mor. of sch is <u>compactifiable</u>

if $\exists$ $X \overset{\text{open}}{\hookrightarrow} \overline{X}$, $f = \bar{f} \circ j$, $\bar{f}$ proper

Eg $X \to \operatorname{Spec}(k)$ quasi-proj.

$\rightsquigarrow$ For such f, chose $\bar{f}$ and let $j: X \hookrightarrow \bar{X}$

$R^n f_! := R^n \bar{f}_* \circ j_! : \{$constr. $\mathbb{Q}_\ell$-sheaves on $X\} \to \{$constr. $\mathbb{Q}_\ell$-sh. on $S\}$

"derived push-forward with cpct support"

For $X \to \mathrm{Spec}(k)$ var. ($k = \bar{k}$) $H^n_c(X, F) := R^n f_!(F)$ for F constr. on $\mathbb{Q}_\ell$

Fact: This is ind. of the choice of compactif.

$\rightsquigarrow$ · $H^0_c(X,F) = \{ s \in \Gamma(F) \mid \mathrm{supp}(s) \text{ is proper} \}$

· The $H^n_c(X,F)$ are f-dim $/k$

· $\underbrace{H^i_c(X, \mathbb{Q}_\ell) \times H^{2d-i}(X, \mathbb{Q}_\ell)(d) \to \mathbb{Q}_\ell}_{\text{non-deg.}}$ P.D.

For X/k proper $Z \subset X$ cl. subsch, $U = X \setminus Z$

F constr. $\mathbb{Q}_\ell$-sheaf on X $\rightsquigarrow$

$\to H^i_c(U, F|_U) \to H^i(X,F) \to H^i(Z, F|_Z) \to H^{i+1}_c(U, F|_U) \to$

Thm $X/\mathbb{F}_q$ $\dim X = d$ even

$\rightsquigarrow$ Every e.v. γ on $F^* \circlearrowright H^d(\bar{X}, \mathbb{Q}_\ell)$

is alg. and every complex abs. val of γ satisfies

$$|\gamma| \leq q^{d/2 + 1/2}$$

Pf: · by ind. on d $d=0$ ok

· can always $X/\mathbb{F}_q$ by $X_{\mathbb{F}_{q^e}}/\mathbb{F}_{q^e}$ for some $e \geq 1$

Take $X \xleftarrow[\text{birational}]{} \tilde{X}$ Lefsch. pencil
$\downarrow f$
$D = \mathbb{P}^1 \supset A$ ← finite set over which f is sing.

- wlog everything def / $\mathbb{F}_q$
- $H^i(\bar{X}, \mathbb{Q}_\ell) \hookrightarrow H^i(\bar{\tilde{X}}, \mathbb{Q}_\ell)$:

⇝ injectivity for H^i follows from P.D.

$$H^i(\bar{X}, \mathbb{Q}_\ell) \times H^{2d-i}(\bar{X}, \mathbb{Q}_\ell) \to H^{2d}(\bar{X}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow \cong$$
$$H^i(\bar{\tilde{X}}, \mathbb{Q}_\ell) \times H^{2d-i}(\bar{\tilde{X}}, \mathbb{Q}_\ell) \to H^{2d}(\bar{\tilde{X}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$$

$H^i(\bar{X}, \mathbb{Q}_\ell) \to H^i(\bar{X}, \mathbb{Q})$ comp. with F^*
$\rightsquigarrow$ enough to look at $H^i(\bar{\bar{X}}, \mathbb{Q}_\ell)$

- Have Leray spectral seq:

$$E_2^{p,q} = H^p(\bar{D}, R^q f_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\bar{\bar{X}}, \mathbb{Q}_\ell)$$

i.e: For $i \geq 0$ $\exists$ nat. filtration on $H^i(\bar{\bar{X}}, \mathbb{Q}_\ell)$ the subquot. of which are isomorphic to $E_2^{p,q}$ for all p,q s.t. $p+q=i$.

This is comp. with F^*

$\Rightarrow$ to understand $F^* \curvearrowright H^d(\bar{X}, \mathbb{Q}_\ell)$ it's

understand $F^* \curvearrowright E_2^{p,q}$ for all $p+q=d$

- $E_2^{p,q} = H^p(\underset{\mathbb{P}^1}{\underset{\cap}{\Delta}}, R^{d-p} f_* \mathbb{Q}_\ell)$

$\cdot \; \cancel{= 0 \text{ for } p > 2}, \; p < 0$

$\rightsquigarrow E_2^{0,d}, E_2^{1,d-1}, E_2^{2,d-2}$

$$E_2^{2,d-2} = H^2(\bar{D}, R^{d-2} f_*)$$

$$= H^2(\bar{D}, R^{n-1} f_* \mathbb{Q}_\ell)$$

$d-1 = n$
$= 2m+1$

Saw: $R^{n-1} f_* \mathbb{Q}_\ell$ is cst on D

with fiber in α $H^{n-1}(\bar{X}_\alpha, \mathbb{Q}_\ell)$

$\alpha : \mathrm{Spec}(\bar{\mathbb{F}}_p) \to D \setminus A$

$$E_2^{2,d-2} \cong H^2(\bar{D}, \mathbb{Q}_\ell) \otimes H^{n-1}(\bar{X}_\alpha, \mathbb{Q}_\ell)$$

$\circlearrowleft$ F^* has e.v q

$\circlearrowleft$ F^*

To apply ind. need an even-dim var

$Y \subset X^{\alpha}$ gen hyperplane sect.

$H^{n-1}(\bar{X}_q, \mathbb{Q}_\ell) \hookrightarrow H^{n-1}(Y, \mathbb{Q}_\ell)$ ← weak Lefschetz thm

comp. with F^{α} ⇝ $\dim Y = \dim X - 2$

by ind. e.v γ of $F^{\alpha} \curvearrowright H^{n-1}(X, \mathbb{Q}_\ell)$ are alg

$$|\gamma| \le q^{\frac{d-2}{2} + \frac{1}{2}}$$

To get e.v on $E_2^{?,d-2}$ multiply by q. ⇝ ok

- $E_2^{0,d} = H^0(\bar{D}, R^{n+1} f_* \mathbb{Q}_\ell)$ $\qquad j : D \setminus A \hookrightarrow D$
- E space of vanishing cycles

$E = 0$: $\quad 0 \to \bigoplus_{s \in A} \mathbb{Q}_\ell(n-m)_s \to R^{n+1} f_* \mathbb{Q}_\ell \to \underbrace{j_* j^* R^{n+1} f_* \mathbb{Q}_\ell}_{\text{cst on } D} \to 0$

$H^0(\bar{D}, -) \rightsquigarrow$

$$0 \to \bigoplus_{s \in A} \mathbb{Q}_\ell(n-m) \to E_2^{0,d} \to H^0(\bar{D}, \mathbb{Q}_\ell) \oplus H^{n+1}(\bar{X}, \mathbb{Q}_\ell) \to 0$$

First look at this; F^* acts by $q^{n-m} = q^{d/2}$

[Aside Geom vs arith. Frob:

$$X \ / \mathbb{F}_q \ \rightsquigarrow \ \bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$$

$$\circlearrowright$$

$$\bar{F} = F \times \mathrm{id}_{\bar{\mathbb{F}}_q} \leftarrow \text{geom. Frob}$$

$$\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}_q / \mathbb{F}_q) \quad x \mapsto x^q \leftarrow \text{arith. Frob}$$

$$\rightsquigarrow \ \mathrm{id}_X \times \sigma : \bar{X} \to \bar{X} \quad (\text{not over } \bar{\mathbb{F}}_q)$$

$$\bar{F}^*, \ (\mathrm{id}_X \otimes \sigma)^* : H^i(\bar{X}, \mathbb{Q}_\ell) \circlearrowright$$ are inverse to each other

Why: $F \circ (\mathrm{id} \circ \sigma) : \bar{X} \to \bar{X}$ abs. Frob
($x \mapsto x^q$ on rings of functions)

acts as id on $H^i(\bar{X}, \mathbb{Q}_\ell)$

$\exists$ version of this for $X/\mathbb{F}_q$ + $\mathcal{F}$ sheaf on $X/\mathbb{F}_q$

$\mathbb{Q}_\ell(1) \leftrightarrow (\mu_{\ell^n})_{n \geq 1} \otimes \mathbb{Q}_\ell / \mathbb{F}_q$

$\rightsquigarrow F^* \circlearrowright \mathbb{Q}_\ell(1)$ by q^{-1}

Back to pf:

$E=0$ $\quad 0 \to \bigoplus_{s\in A} \mathbb{Q}_\ell(n-m) \to E_2^{a,d} \to H^0(D, \mathbb{Q}_\ell) \otimes H^{n+1}(\bar{X}_a, \mathbb{Q}_\ell) \to 0$

$$F^* = q^{n-m} = q^{d/2}$$

do this by ind. or P.D

$E \neq 0$: $R^{n+1} f_* \mathbb{Q}_\ell$ cst with fibre $\rightsquigarrow$ GL $H^{n+1}(\bar{X}, \mathbb{Q}_\ell)$

$\rightsquigarrow$ do as before.

- $E_2^{1,d-1} = H^1(\bar{D}, R^n f_* \mathbb{Q}_\ell)$ $\quad E \subset V$ $\quad V \times V \to \mathbb{Q}_\ell$ P.D
- $E = 0 \to R^n f_* \mathbb{Q}_\ell$ cst $\Rightarrow H^1(\bar{D}, \ldots) = 0$
- $E \neq 0$: $\underbrace{E \subset E^\perp}_{\text{omit}}$ or $E \not\subset E^\perp$ $\quad j : D \setminus A \hookrightarrow D$

$\hookrightarrow \quad 0 \to A \to R^n f_* \mathbb{Q}_\ell \to B \to 0$ $\quad B, C$ are cst

$0 \to C \to A \to j_*(\mathcal{G}) \to 0$ not cst in gen

$\mathcal{G} \leftrightarrow \pi_1^t(D \setminus A, \omega) \curvearrowright E / E \cap E^\perp$

loc cst on $D \setminus A$

$H^1(D, \mathbb{Q}_\ell) = 0$

$$H^1(\bar{D}, A) \to H^1(\bar{D}, R^1 f_* \mathbb{Q}_\ell) \to 0$$

$$0 \to H^1(\bar{D}, A) \to H^1(\bar{D}, j_* \mathcal{G})$$

comp with Frob $\rightsquigarrow$ look at $H^1(\bar{D}, j_* \mathcal{G})$

- $A = \emptyset \Rightarrow j_* \mathcal{G}$ cst on $D \rightsquigarrow = 0$
- $S = D \setminus A$ affine curve

look at this $\to H^1_c(\bar{S}, \mathcal{G}) \to H^1(\bar{D}, j_*(\mathcal{G})) \to H^1(\bar{A}, \mathcal{G}|_A)$ $= 0$, $\dim A = 0$

$\rightsquigarrow$ Need to prove statement for $F^\times \circlearrowright H^1_c(S, \mathcal{F})$

where $\mathcal{F} \leftrightarrow \pi_1^+(S, \omega) \circlearrowright M = E/_{E \cap E^\perp}$

$\psi : M \times M \to \mathbb{Q}_\ell$ non-deg. alternating

$\varrho : \pi_1^+(S, \omega) \to Sp(\psi)(\mathbb{Q}_\ell)$

$$W = \{ \Theta \in M \mid N(\Theta) : \underset{M \to M}{x \mapsto \Psi(x,\Theta)\Theta} \in \mathfrak{g} \}$$

Want to show $W = M$:

$$E = \sum_{\substack{s \in A \\ g \in \pi_1^+(s,\omega)}} g \cdot \delta_s$$