S var $/ \mathbb{F}_q$

$\mathcal{G}/S$ constr. $\mathbb{Q}_\ell$-sheaf

$s \in |S|$ cl. pt $\qquad \mathcal{G} = \mathbb{Q}_\ell \rightsquigarrow \dfrac{1}{(1 - T^{\deg(s)})}$

$\mathcal{G}_s$ $\mathbb{Q}_\ell$-sheaf $/ \operatorname{Spec}(k(s))$

$\leftrightarrow \mathcal{G}_{\overline{k(s)}}$ f. dim v.s $/ \mathbb{Q}_\ell$

$+ \; f_s \curvearrowright \mathcal{G}_{\overline{k(s)}}$

action of $\sigma \in \operatorname{Gal}(\overline{k(s)}/k(s))$ ar. Frob.

$\rightsquigarrow \zeta_{\mathcal{G}}(T) =$

$$= \prod_{s \in |S|} \det\left(1 - T^{\deg(s)} f_s^{-1} \,\middle|\, \mathcal{G}_{\overline{k(s)}}\right)^{-1} \in \mathbb{Q}_\ell[[T]]$$

<u>Thm</u> $\zeta_{\mathcal{G}}(T) = \prod_{i=0}^{2\dim(X)} \det(1 - TF^* | H^i_c(\bar{S}, \bar{\mathcal{G}}))^{(-1)^{i+1}}$.

$\bar{\mathcal{G}} = (\bar{S} \to S)^* \mathcal{G}$ $\qquad F : \bar{S} \to \bar{S}$

$\mathcal{G}$ def $\mathbb{F}_q \rightsquigarrow F_{\mathcal{G}} : F^* \bar{\mathcal{G}} \xrightarrow{\cong} \bar{\mathcal{G}}$

$\rightsquigarrow F^* : H^i_c(\bar{S}, \bar{\mathcal{G}}) \to H^i_c(\bar{S}, F^* \bar{\mathcal{G}}) \xrightarrow{F_{\mathcal{G}}} H^i_c(\bar{S}, \bar{\mathcal{G}})$

<u>Rmk</u>: Assume $\mathcal{G}$ lisse

- For S affine $H^i_c(\bar{S}, \bar{\mathcal{G}}) = 0$ for $0 \leq i < d$

· P.D $\mathcal{G} \rightsquigarrow \mathcal{G}^\vee = \mathrm{Hom}(\mathcal{G}, \mathbb{Q}_\ell)$ $d = \dim S$

$$H^i_c(\bar{S}, \bar{\mathcal{G}}) \times H^{2\dim(X)-i}(\bar{S}, \bar{\mathcal{G}}^\vee)(d) \to \mathbb{Q}_\ell$$

non-deg.

<u>Situation:</u> $S = \mathbb{P}^1 \setminus A$ affine curve $/\mathbb{F}_q$

$\mathcal{G}/S \leftrightarrow \pi_1^{t}(S,\omega) \curvearrowright E/_{E\cap E^\perp}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad M$

$\psi : M \times M \to \mathbb{Q}_\ell$

<u>Want:</u> $F^* \curvearrowright H^1_c(\bar{S}, \bar{\mathcal{G}})$ has alg. e.v λ, all complex norms satisfy $|\lambda| \le q^{d/2+1}$

$\overline{}$

$1 \to \pi_1(\bar{S},\omega) \to \pi_1(S,\omega) \to \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1$

$$\begin{array}{ccc} & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) & \\ & \wr\shortparallel & \in \\ d \longrightarrow & \hat{\mathbb{Z}} & 1 \end{array}$$

$\sigma \in \pi_1^+(S, \omega)$ $\quad x, y \in M$:

$$\psi(\sigma x, \sigma y) = q^{-d \cdot d(\sigma)} \psi(x, y)$$

$$\rightsquigarrow \quad \pi_1^+(\bar{S}, \omega) \xrightarrow{\rho} Sp(\psi)(\mathbb{Q}_\ell)$$

Thm (Kazhdan-Margulis):
Im ρ is open in $Sp(\psi)(\mathbb{Q}_\ell)$

Pf: $G = \operatorname{Im} \rho$ ℓ-adic Lie group
$\rightsquigarrow \mathfrak{g}$ Lie algebra
Want $\mathfrak{g} = \operatorname{Lie} Sp(\psi)$

$$\mathfrak{g} \subset \mathrm{Lie}\,Sp(\psi) \subset \mathrm{Lie}\,GL(M)$$

$$\mathrm{Lie}\,Sp(\psi) = \{\varphi : M \to M \mid \forall x,y,\ \psi(\varphi x, y) + \psi(x, \varphi x) = 0\}$$

$$\mathrm{Lie}\,GL(M) = \mathrm{End}(M)$$

$$W := \{\theta \in M \mid \underset{N(\theta):\, x \mapsto \psi(x,\theta)\theta}{M \to M} \in \mathfrak{g}\}$$

Want: $W = M$

$s \in A \rightsquigarrow \delta_s \in M$ van. cycle

$\gamma_s \in \pi_1^t(S, \omega) : \gamma_s(x) = x \pm \psi(x, \delta_s)\delta_s$

$G \xrightarrow{\log} \mathfrak{g} : \gamma_s \mapsto \pm N(\gamma_s) \Rightarrow \delta_s \in W$

- Have: $\mathbb{Q}_\ell \cdot W \subset W$

  $u, v \in W$: $\psi(u,v) = 0$ or $u+v \in W$: $\overset{(*)}{}$

$$\psi(u,v) N(u+v) = \psi(u,v)(N(u)+N(v)) + [N(u), N(v)] \in \mathfrak{g}$$

- Take $0 \neq U \subset W$ maxima $\mathbb{Q}_\ell$-v.s.
- Let $\theta \in W$ which is not orthogonal to all of U

  $\Rightarrow (U \setminus (U \cap \theta^\perp)) + \mathbb{Q}_\ell \theta \underset{(*)}{\subset} W$

  $W \subset M$ Zar. closed

$U \setminus (U \cap \Theta^{\perp}) \subset U$ Zar. dense

$+ \mathbb{Q}_\ell \Theta \qquad + \mathbb{Q}_\ell \qquad - " -$

$\Rightarrow U + \mathbb{Q}_\ell \Theta \subset W \Rightarrow \Theta \in U$

$\Rightarrow W = U \cup (U^{\perp} \cap W)$

$\Rightarrow U$ is stable under

$x \mapsto x + \lambda \psi(x, \theta) \Theta$

for $\Theta \in W$, $\lambda \in \mathbb{Q}_\ell$

Such operations generate $\pi_1^{+}(\bar{S}, \omega)$ top.

$\Rightarrow U \subset M$ $\pi_1^{+}(\bar{S}, \omega)$-inv.

M irr under $\pi_1^{+}(\bar{S}, \omega) \Rightarrow U = M = W$

i.e $\forall\, \theta \in M,\ N(\theta) \in \mathfrak{g}$.

<u>Fact</u>: The $N(\theta)$ for $\theta \in M$ generate the Lie alg $\operatorname{Lie} Sp(\psi)$

$\square$

$$\zeta_{\mathcal{G}}(T) = \prod_{s \in |S|} \det(1 - f_s^{-1} T)^{-1}$$

$$= \frac{\det(1 - F^* T \mid H^1_c(\bar{S}, \bar{\mathcal{G}}))}{\det(1 - F^* T \mid H^2_c(\bar{S}, \bar{\mathcal{G}}))}$$

We will show: $\forall s$: $\det(1 - f_s^{-1} T) \in \mathbb{Q}[T]$

Assume this for now.

Claim: For $i=1,2$ $\det(1-F^* T \mid H^i_c(\ldots)) \in \bar{\mathbb{Q}}[T]$

Pf: By assumption, above, enough to consider $i=2$.

$$\text{P.D:} \quad H^2_c(\bar{S}, \bar{\mathcal{G}}) = H^0(\bar{S}, \bar{\mathcal{G}}^\vee)^\vee \otimes \mathbb{Q}_\ell(-1)$$

$$\pi = \pi_1(\bar{S}, \omega) : H^0(\bar{S}, \bar{\mathcal{G}}) = M^\pi \quad H^0(\bar{S}, \bar{\mathcal{G}}^\vee) = (M^\vee)^\pi$$

$$H^0(\bar{S}, \bar{\mathcal{G}}^\vee)^\vee = \left((M^\vee)^\pi\right)^\vee = M_\pi = M/\{\sigma m - m \mid \sigma \in \pi, m \in M\}$$

$s \in |S|$: $\omega : \mathrm{Spec}(\Omega) \to S$

choose $\overline{k(s)} \hookrightarrow \Omega$

$\leadsto \quad \mathrm{Gal}\left(\overline{k(s)}/k(s)\right) \to \pi_1^{t}(S,\omega)$

$\quad G_s \text{ arith. Frob.} \mapsto \Phi_s$ det. up to conj.

$f_s \curvearrowright \mathcal{G}_{\overline{k(s)}} = \Phi_s \curvearrowright M$

$$\begin{array}{ccc} M & \to & M^{\pi} \\ \circlearrowleft & & \circlearrowleft \\ f_s = \Phi_s & \to & (\mathbb{F}^{*})^{\deg s} \end{array}$$

By above assumption, these have alg. e.v. □(claim)

- By applying this arg. to g^{ah} for $h \geq 1$ get: All e.v of $F^* \tilde{C} (M^{\otimes h})_\pi$ are also alg.

<u>Claim</u>: These ev $\uparrow$ for $h \geq 1$ have all cplx norms $= |q^{hd/2}|$

<u>Pf</u>: $1 \to Sp(\psi) \to A(\psi) \xrightarrow{g \mapsto \lambda} G_m \to 1$

$A(\psi) = \{ g \in GL(M) \mid \exists \lambda : \psi(g-, g-) = \lambda \psi(-,-) \}$

$a \in \mathbb{Q}_\ell^* \to \underset{m \mapsto am}{M \to M} \in A(\psi) \mapsto a^2$

$A_0 \subset A(\psi)(\mathbb{Q}_\ell)$ subgr gen by $Sp(\psi)(\mathbb{Q}_\ell)$ and $m \mapsto am, a \in \mathbb{Q}_\ell^*$

$$A(\psi)(\mathbb{Q}_\ell)/A_0 \hookrightarrow \mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^2$$ ← finite

$\Rightarrow$ A_0 is of fin. index in $A(\psi)(\mathbb{Q}_\ell)$

$\Pi = \pi_1(\bar{S}, \omega) \subset Sp(\psi)(\mathbb{Q}_\ell)$ open

$\Rightarrow$ Π is dense in $Sp(\psi)$

$$A(\psi)(\mathbb{Q}_\ell)/Sp(\psi)(\mathbb{Q}_\ell) \quad C(M^{an})_\Pi$$

$\subset \mathbb{Q}_\ell^\times$ abelian

λ_0 e.v. of $\Phi_s \curvearrowright (M^{\otimes 4})_\pi$

$\rightsquigarrow \exists \lambda : A(\psi)(\mathbb{Q}_\ell)/Sp(\psi)(\mathbb{Q}_\ell) \to \overline{\mathbb{Q}}_\ell^\times$ char

$\lambda(\Phi_s) = \lambda_0$

For $g \in$, $\lambda(g)$ is an ev of $g \curvearrowright (M^{\otimes 4})_\pi$

Have $\chi : A(\psi)(\mathbb{Q}_\ell)/Sp(\psi)(\mathbb{Q}_\ell) \to \mathbb{Q}_\ell^\times$

$g \mapsto \det(g \curvearrowright (M^{\otimes 4})_\pi)$

$A_0 \subset A(\psi)(\mathbb{Q}_\ell)$

$\rightsquigarrow A_0 / Sp(\psi)(\mathbb{Q}_\ell) \subset A(\psi)(\mathbb{Q}_\ell) / Sp(\psi)(\mathbb{Q}_\ell)$

$\lambda^{\dim M}$, χ'^{k} $\quad$ gen $\begin{matrix} M \to M \\ m \mapsto am \end{matrix}$ $\quad a \in \mathbb{Q}_\ell^\times$ agree on A_0

$A_0 \subset A(\psi)(\mathbb{Q}_\ell)$ finite index

$\Rightarrow (\lambda^{\dim V} \chi^{-k})$ has finite order

$$\Rightarrow |\iota^h(\phi_s)| = |\iota^h(\phi_s)|$$

$$= \Big| \det\big(\underbrace{\phi_s\, C^{\cdot}\, (M^{\otimes h})_{\pi}}_{q^{-d h d(\phi_s)/2}\,(?.)} \big) \Big|$$

$n = \dim M$

$$= q^{-n \cdot d \cdot h \deg(s)/2}$$

<u>Claim</u>: For $s \in |S|$, the e.v. of $\phi_s \circlearrowright M$ are alg. numbers all of whose complex norms are $= q^{-d\deg(s)/2}$

<u>Pf</u>: $\zeta_{\mathcal{G}^{\otimes n}}(T) = \prod_{s \in |S|} \underbrace{\det(1 - \phi_s^{-1} T^{\deg(s)} | M^{\otimes n})^{-1}}_{P_s \in \mathbb{Q}[[T]] \text{ with non-neg. coeff.}}$

$n \geq 2$ even

$$\det(1 - \tau T)^{-1} = \exp\left(\sum_{n \geq 1} \mathrm{Tr}(\tau^n) T^n / n\right)$$

$$\mathrm{Tr}(\tau \otimes \tau) = \mathrm{Tr}(\tau)^2$$

$\Rightarrow$ rad of conv (around 0) of P_s

$\geq$ rad. of conv. of $\mathfrak{Z}_{gan}$:

$\mathfrak{Z}_{gan}(T) = \sum a_i T^i \qquad P_s = \sum b_{s,i} T^i$

$\limsup_{i \to \infty} |a_i|^{1/i} \qquad\qquad a_i = \sum_{j_1 + \dots + j_r = i} \prod b_{s_e, j_e}$

$\geq \limsup_{i \to \infty} |b_{s,i}|^{1/i}$

rad of conv of $P_s \geq$ — " — $\mathfrak{J}_{S^{\otimes k}}$

$\geq q^{-(1+kd)/2}$
above claim

λ e.v of $\Phi_s \curvearrowright M$

$\rightsquigarrow \lambda^k$ e.v. of $\Phi_S \curvearrowright M^{\otimes k}$

$\Rightarrow |\lambda|^k \geq q^{-(1-kd)/2 \cdot \deg(s)}$

$\overset{k\to\infty}{\Rightarrow} |\lambda| \geq q^{-d \deg(s)/2}$

Do the same for $M^\vee$ $\rightsquigarrow$ " = "

$\square$ (claim)

Claim All ev of $F^* \circlearrowright H^1_c(\bar{S}, \bar{\mathcal{G}})$ have cplx norms $\le q^{a/2+1}$

Pf: $Z_{\mathcal{G}}(T)^{-1} = \prod_{s \in |S|} \det(1 - T^{\deg(s)} \phi_s^{-1} \mid M)$

$s \in |S|$: $\lambda_1(s), \dots, \lambda_n(s)$ e.v of $\phi_s^{-1} \circlearrowright M$

$= \prod_{s \in |S|} (1 - \lambda_1(s) T^{\deg(s)}) \cdots (1 - \lambda_n(s) T^{\deg(s)})$

Converges if
$$\sum_{1\le i\le n}\ \sum_{s\in|S|} |\chi_i(s)|\,T^{\deg(s)} \text{ converges}$$

claim above $\Rightarrow$
$$= n \sum_{s\in|S|} q^{d\,\deg(s)/2}\,T^{\deg(s)}$$

$$\exists C : \{s\in|S| \mid \deg(s)=m\} \le C\cdot q^m$$

$$\le C \sum_m \left(q^{d/2+1}\,T\right)^m \quad \text{converges for } |T|<q^{-(d/2+1)}$$

poles have to be outside $\Rightarrow$ estimate. □