$\mathcal{G} / S = \mathbb{P}^1 \setminus A \rightsquigarrow \pi_1(S,\omega) \curvearrowright E/_{E \cap E^\perp} = M$

$\alpha \in |S| \rightsquigarrow \phi_\alpha \in \pi_1(S,\omega)$ (up to conj.)

Want: $\forall \alpha \in |S|: \det(1 - \phi_\alpha^{-1} T) \in \mathbb{Q}[T]$

$$1 \to \pi_1(\bar{S},\omega) \to \pi_1(S,\omega) \xrightarrow{d} \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1, \quad \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$$

For $\pi_1(S,\omega) \circlearrowright V$, $\alpha \in |S|$ $\chi_V(\alpha) := \det(1 - \phi_\alpha^{-1} T | V)$

Claim For $\pi_1(S,\omega) \curvearrowright V$ s.t $\pi_1(\bar{S},\omega)$ acts trivially

$\exists\ \lambda_1, \ldots, \lambda_r \in \bar{\mathbb{Z}}_\ell^*$ s.t.

$\forall \alpha \in |S|\quad \chi_V(\alpha) = \prod_{i=1}^{r} \left(1 - \lambda_i^{-d(\alpha)} T\right)$

Here: $d(\alpha) = \delta(\phi_\alpha) = \deg(\alpha)$

Pf: $\pi_1(S,\omega) \curvearrowright V$ $\rightsquigarrow$ take λ_i = e.v of $\sigma \in \hat{\mathbb{Z}}$ arith. Frob.

$\downarrow$

$\hat{\mathbb{Z}} \curvearrowright$

Rmk: $\mathbb{Z} \curvearrowright \bar{\mathbb{Z}}_\ell^*$ extends to $\hat{\mathbb{Z}} \curvearrowright \bar{\mathbb{Z}}_\ell^*$

$(1, x \mapsto) \ x^n$

Claim: $\exists \lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s \in \bar{\mathbb{Z}}_\ell^\times$ s.t.:

$\forall i, j$: $\mu_i \neq \lambda_j$

$\forall \alpha \in |S|$: $R_\alpha(T) = \dfrac{\prod (1 - \lambda_i^{-d(\alpha)} T)}{\prod (1 - \mu_j^{-d(\alpha)} T)} \cdot \chi_\mu(\alpha) \in \mathbb{Q}(T)$

Pf: Have: $0 \to \mathcal{A} \to R^1 f_* \mathbb{Q}_\ell \to \mathcal{B} \to 0$ $\qquad f \downarrow$ X over $\mathbb{P}^1$

lisse $\mathbb{Q}_\ell$-sh. on S: $0 \to \mathcal{C} \to \mathcal{A} \to \mathcal{G} \to 0$

Pf: $\chi_M(\alpha) = \dfrac{\chi_A(\alpha)}{\chi_C(\alpha)} = \dfrac{\chi_B(\alpha)}{\chi_C(\alpha)} \dfrac{1}{\chi_{R^n f_* \mathbb{Q}_\ell}(\alpha)}$

$$\prod_{i=0}^{2d} \chi_{R^i f_* \mathbb{Q}_\ell}(\alpha)^{(-1)^{i+1}} = \mathfrak{Z}_{\mathcal{X}_\alpha}(T) \in \mathbb{Q}(T)$$

A, C, $R^i f_* \mathbb{Q}_\ell$ for $i \neq n$ become cst on $\mathfrak{J}$

conclude by first claim $\square$ (claim)

$g \in \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}) \curvearrowright \bar{\mathbb{Q}}_\ell[[T]]$

$$\rightsquigarrow \frac{\chi_M(\alpha) \prod (1-\lambda_i^{-d(\alpha)}T) \prod (1-g(\mu_j)^{-d(\alpha)}T)}{\prod (1-\mu_j^{-d(\alpha)}T)}$$

$$= \prod (1-g(\lambda_i)^{-d(\alpha)}T) \cdot g(\chi_M(\alpha)) \in \bar{\mathbb{Q}}_\ell[T]$$

Want to show first: $\prod (1-\mu_j^{-d(\alpha)}T) \in \mathbb{Q}[T]$

$\pi_1(S,\omega)$ compact $\rightsquigarrow$ $\exists$ Haar measure

Will show: $\exists\ F \subset \pi_1(S,\omega)$ closed, measure 0

inv. under conj s.t.:

$\forall \sigma \in \pi_1(S,\omega) \setminus F : \forall j : \mu_j^{d(\sigma)}$ is not an e.v.

of $\sigma : M \to M$

Then: For α s.t. $\phi_\alpha \notin F$ $\prod_j (1 - \mu_j^{-d(\alpha)} T)$

is coprime to $\chi_M(\alpha)$

$E \subset \mathbb{Z}^{>1}$: $E = \{ m > 1 \mid \exists i,j \ (\lambda_i/\mu_j)$ is a root of unity of order $m \}$

$\uparrow$

finite

$\cup \{ m > 1 \mid \exists i,j$: $\mu_i \neq g(\mu_j)$, $\mu_i/g(\mu_j)$ is a root of unity of order $m \}$

$\bigcup_{m \in E} m\hat{\mathbb{Z}} \subsetneq \hat{\mathbb{Z}}$ closed

$\rightsquigarrow U = d^{-1}\left(\hat{\mathbb{Z}} \setminus \bigcup_{m\in E} m\hat{\mathbb{Z}}\right) \subset \pi_1(S,\omega)$ open non-empty

For α s.t. $F_\alpha \in U$:

$\prod(1-\lambda_i^{-d(\alpha)}T)$ and $\prod(1-\mu_j^{-d(\alpha)}T)$

are coprime

Assume: $d \cdot \pi_1(S,\omega) \twoheadrightarrow \hat{\mathbb{Z}}$

$\rightsquigarrow \quad U \setminus (F \cap U) \subset \pi_1(S, \omega)$ open $\rightarrow$ pos. Haar measure

inv under conj.

Chebotarev density thm $\Rightarrow \quad \exists \alpha \in |S| : \, F_\alpha \in U \setminus (F \cap U)$

$\Rightarrow$ For such α: $\quad \prod_j (1 - \mu_j^{-d(\alpha)} T) = \prod_j (1 - g(\mu_j)^{-d(\alpha)} T)$

second part of ξ $\Rightarrow \quad \prod_j (1 - \mu_j T) = \prod (1 - g(\mu_j) T) \in \mathbb{Q}[T]$

Same arg $\Rightarrow$ $\prod(1-\lambda_i T) \in \mathbb{Q}[T]$

$$\Rightarrow \forall \alpha : \chi_M(\alpha) \in \mathbb{Q}[T] \quad (\boxtimes)$$

Still have to construct F!

For $\alpha \in \bar{\mathbb{Q}}_\ell^\times$: $F_\alpha = \left\{ \sigma \in \pi_1(S,\omega) \,\middle|\, \begin{array}{l} \alpha^{d(\sigma)} \text{ is an ev} \\ \text{of } \sigma : M \to M \end{array} \right\}$

closed, inv under conj.

<u>Want</u>: F_α has measure zero

Pf $\nearrow$:

$$1 \to Sp(\psi)(\mathbb{Q}_\ell) \to A(\psi)(\mathbb{Q}_\ell) \xrightarrow{\chi} \mathbb{Q}_\ell^\times \text{ or } \mathbb{Z}_\ell^\times$$

d = fiber dim X + $\frac{1}{\mathbb{P}^1}$

$$\begin{array}{ccc} & & \uparrow \\ \pi_1(S,\omega) & \longrightarrow & G = \{(m,\sigma) \in \mathbb{Z} \times A(\psi)(\mathbb{Q}_\ell) \mid \chi(\sigma) = q^{dm}\} \\ \sigma & \mapsto & (d(\sigma), \sigma|_M) \end{array}$$

has open image: $\cdot$ $\pi_1(\bar{S},\omega)$ has open image in $Sp(\psi)(\mathbb{Q}_\ell)$

$\cdot$ $d : \pi_1(S,\omega) \twoheadrightarrow \mathbb{Z}$

$F_\alpha \subset \tilde{F}_\alpha = \{ (n, \sigma) \in G \mid \alpha^n$ is an ev of $\sigma : M \to M \}$

Enough to show : $\tilde{F}_\alpha$ has measure zero

$\ulcorner$ G is locally cpct $\rightsquigarrow \exists$ Haar measure

Fix $m \in \Sigma_\ell$

$\overline{\mathbb{Q}}_\ell$ pts of fiber of G / m

H

$$\tilde{F}_\alpha \cap H \subset M = \left\{ g \in H \;\middle|\; g \text{ and its } \overline{\mathbb{Q}}_\ell\text{-conj. have } \alpha^m \text{ as e.v.} \right\} \subset \left\{ g \in A(\psi)(\overline{\mathbb{Q}}_\ell) \;\middle|\; \chi(g) = q^{dm} \right\}$$

simply trans

$Sp(\psi)(\overline{\mathbb{Q}}_\ell)$

Zariski closed subset not all of H

$\rightsquigarrow$ H is a sm. abs. irr. var / $\overline{\mathbb{Q}}_\ell$

has natural measure from Haar measure on $\overline{\mathbb{Q}}_\ell$

proper cl. subs. have measure zero

measures are comp $\rightsquigarrow$ $\tilde{F}_\alpha \cap \{m\} \times A(\psi)(\overline{\mathbb{Q}}_\ell)$ has meas. zero.

Fubini $\Rightarrow F_\alpha$ has measure zero.

$\square$ (Weil conj)

Weil II:

Fix: $\tau: \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$

Def: X var. $/\mathbb{F}_q$ $\mathcal{G}/X$ constr. $\overline{\mathbb{Q}}_\ell$-sheaf

$\beta \in \mathbb{R}$

- $\mathcal{G}$ is τ-pure of weight β, if $\forall x \in |X|$ all e.v. α of $\phi_x \curvearrowright \mathcal{G}_x$ satisfies $|\tau(\alpha)| = |k(x)|^{\beta/2}$

- $\mathcal{G}$ is pure of wt β, if it's ι-pure of wt β $\forall \iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$.
- $\mathcal{G}$ is ι-mixed if $\exists$ filtr.

$$0 = \mathcal{G}^0 \subset \mathcal{G}^1 \subset \dots \subset \mathcal{G}^r = \mathcal{G}$$

s.t. all $\mathcal{G}^i / \mathcal{G}^{i-1}$ are ι-pure of some wt.

- $\mathcal{G}$ is mixed if $\exists$ filtr. as above s.t. all $\mathcal{G}^i / \mathcal{G}^{i-1}$ are pure of some wt.

· $X \xrightarrow{f} Y$ mor. of var $/\mathbb{F}_q$

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

$R^n f_! = R^n \bar{f}_* \circ j_! : \left\{ \begin{array}{l} \text{constr. } \mathbb{Q}_\ell\text{-sheaves} \\ \text{on } X \end{array} \right\} \to \left\{ \begin{array}{l} \text{constr. } \mathbb{Q}_\ell\text{-sh.} \\ \text{on } Y \end{array} \right\}$

<u>Thm</u> $f: X \to Y$, $\mathcal{G}$ constr. $\mathbb{Q}_\ell$-sheaf on X
τ-mixed, largest τ-wt $= \beta$
$\Rightarrow R^i f_! \mathcal{G}$ is τ-mixed, with largest τ-wt $\leq i + \beta$

Cor: $f: X \to Y$ smooth + proper

$\mathcal{G}/X$ lisse $\overline{\mathbb{Q}}_\ell$-sh.

ι-pure of wt β

$\Rightarrow \forall i \geq 0.\ R^i f_* \mathcal{G}$ is ι-pure of wt $\beta + i$.

"pf": use Poincaré duality

This implies the RH for $f: X \to \operatorname{Spec}(k)$

$\mathcal{G} = \overline{\mathbb{Q}}_\ell$

Hard Lefschetz thm

Thm $X / k = \bar{k}$ smooth proj irr var

$\eta \in H^2(X, \mathbb{Q}_\ell(1))$ class of a hyperpl. sect.

$(X \hookrightarrow \mathbb{P}^N \supset H$ hyperpl. $, \eta = cl(X \cap H)$

For $0 < i \leq n = \dim X$

$$H^{n-i}(X, \mathbb{Q}_\ell) \to H^{n+i}(X, \mathbb{Q}_\ell(i))$$
$$c \mapsto \eta^i \cup c$$

is an iso.

$\cdot E \rightsquigarrow E / E \cap E^\perp$
In fact $E \cap E^\perp = 0$
$\Rightarrow$ Hard Lefschetz